

# Boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent

Jianglong WU

*Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, China*

**Abstract:** In this paper, some boundedness for commutators of fractional integrals are obtained on Herz-Morrey spaces with variable exponent applying some properties of variable exponent and BMO function.

**Keywords:** commutator; fractional integral; BMO; Herz-Morrey space with variable exponent

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## 1 Introduction

Function spaces with variable exponent are being watched with keen interest not in real analysis but also in partial differential equations and in applied mathematics because they are applicable to the modeling for electrorheological fluids and image restoration. The theory of function spaces with variable exponent has rapidly made progress in the past twenty years since some elementary properties were established by Kováčik and Rákosník <sup>[1]</sup>. One of the main problems on the theory is the boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces. By virtue of the fine works <sup>[2-11]</sup>, some important conditions on variable exponent, for example, the log-Hölder conditions et al, have been obtained.

The class of the Herz spaces is arising from the study on characterization of multipliers on the classical Hardy spaces. And the homogeneous Herz-Morrey spaces  $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$  coordinate with the homogeneous Herz space  $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$  when  $\lambda = 0$ . One of the important problems on Herz spaces and Herz-Morrey spaces is the boundedness of sublinear operators. Hernández, Li, Lu and Yang et al <sup>[12-14]</sup> have proved that if a sublinear operator  $T$  is bounded on  $L^p(\mathbb{R}^n)$  and satisfies the size condition

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy$$

for all  $f \in L^1(\mathbb{R}^n)$  with compact support and a.e.  $x \notin \text{supp } f$ , then  $T$  is bounded on the homogeneous Herz space  $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ . In 2005, Lu and Xu <sup>[15]</sup> established the boundedness for some sublinear operators.

The BMO space and the BMO norm are defined respectively as follows:

$$\text{BMO}(\mathbb{R}^n) = \left\{ b \in L^1_{\text{loc}}(\mathbb{R}^n) : \|b\|_{\text{BMO}(\mathbb{R}^n)} < \infty \right\}, \quad \|b\|_{\text{BMO}(\mathbb{R}^n)} = \sup_{B:\text{ball}} \frac{1}{|B|} \int_B |b(x) - b_B| dx.$$

The fractional integral  $I_\beta$  is defined by  $I_\beta(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy$ , the commutator for fractional integral is defined by  $[b, I_\beta]f(x) = b(x)I_\beta(f)(x) - I_\beta(bf)(x)$ , and  $m$ -order commutator for fractional integral is defined by

$$I_{\beta,b}^m(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)(b(x) - b(y))^m}{|x-y|^{n-\beta}} dy,$$

where  $0 < \beta < n, b \in \text{BMO}(\mathbb{R}^n), m \in \mathbb{N}$ . It is easy to see, when  $m = 1$ ,  $I_{\beta,b}^m(f)(x) = [b, I_\beta]f(x)$ ; and when  $m = 0$ ,  $I_{\beta,b}^m(f)(x) = I_\beta(f)(x)$ .

Chanillo<sup>[16]</sup> has initially introduced the commutator  $[b, I_\beta]$  with  $b \in \text{BMO}$  and proved the boundedness on Lebesgue spaces with constant exponent. In 2010, Izuki<sup>[17]</sup> generalizes this result to the case of variable exponent and consider the boundedness on Herz spaces with variable exponent.

In 2010, Izuki<sup>[18]</sup> proves the boundedness of some sublinear operators on Herz spaces with variable exponent. And recently Izuki<sup>[19, 20]</sup> also considers the boundedness of some operators on Herz-Morrey spaces with variable exponent.

Motivated by the study on the Herz spaces and Lebesgue spaces with variable exponent, the main purpose of this paper is to establish some boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent. Our main tools are some properties of variable exponent and BMO function. And we also note that our results are the generalizations of main theorems for Izuki<sup>[17, 19]</sup> on Herz space and Herz-Morrey spaces with variable exponent.

Throughout this paper, we will denote by  $|S|$  the Lebesgue measure and by  $\chi_S$  the characteristic function for a measurable set  $S \subset \mathbb{R}^n$ . Given a function  $f$ , we denote the mean value of  $f$  on  $S$  by  $f_S := \frac{1}{|S|} \int_S f(x) dx$ .  $C$  denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index  $1 < q(x) < \infty$ , we denote by  $q'(x)$  its conjugate index, namely,  $q'(x) = \frac{q(x)}{q(x)-1}$ . For  $A \sim D$ , we mean that there is a constant  $C > 0$  such that  $C^{-1}D \leq A \leq CD$ .

## 2 Preliminaries and Lemmas

In this section, we give the definition of Lebesgue and Herz-Morrey spaces with variable exponent, and state their properties. Let  $E$  be a measurable set in  $\mathbb{R}^n$  with  $|E| > 0$ . We first define Lebesgue spaces with variable exponent.

**Definition 2.1.** Let  $q(\cdot) : E \rightarrow [1, \infty)$  be a measurable function.

1) The Lebesgue spaces with variable exponent  $L^{q(\cdot)}(E)$  is defined by

$$L^{q(\cdot)}(E) = \{f \text{ is measurable function} : \int_E \left( \frac{|f(x)|}{\eta} \right)^{q(x)} dx < \infty \text{ for some constant } \eta > 0\}.$$

2) The space  $L_{\text{loc}}^{q(\cdot)}(E)$  is defined by

$$L_{\text{loc}}^{q(\cdot)}(E) = \{f \text{ is measurable function} : f \in L^{q(\cdot)}(K) \text{ for all compact subsets } K \subset E\}.$$

The Lebesgue space  $L^{q(\cdot)}(E)$  is a Banach space with the norm defined by

$$\|f\|_{L^{q(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left( \frac{|f(x)|}{\eta} \right)^{q(x)} dx \leq 1 \right\}.$$

Now, we define two classes of exponent functions. Given a function  $f \in L_{\text{loc}}^1(E)$ , the Hardy-Littlewood maximal operator  $M$  is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r) \cap E} |f(y)| dy \quad (x \in E),$$

where  $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ .

**Definition 2.2.** 1) The set  $\mathcal{P}(\mathbb{R}^n)$  consists of all measurable functions  $q(\cdot)$  satisfying

$$1 < \text{ess inf}_{x \in \mathbb{R}^n} q(x) = q_-, \quad q_+ = \text{ess sup}_{x \in \mathbb{R}^n} q(x) < \infty.$$

2) The set  $\mathcal{B}(\mathbb{R}^n)$  consists of all measurable functions  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfying that the Hardy-Littlewood maximal operator  $M$  is bounded on  $L^{q(\cdot)}(\mathbb{R}^n)$ .

Next we define the Herz-Morrey spaces with variable exponent. Let  $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ ,  $A_k = B_k \setminus B_{k-1}$  and  $\chi_k = \chi_{A_k}$  for  $k \in \mathbb{Z}$ .

**Definition 2.3.** Let  $\alpha \in \mathbb{R}$ ,  $0 \leq \lambda < \infty$ ,  $0 < p < \infty$ , and  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . The Herz-Morrey space with variable exponent  $M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  is defined by

$$M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$

Compare the Herz-Morrey space with variable exponent  $M\dot{K}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  with the Herz space with variable exponent  $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ <sup>[20]</sup>, where

$$\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p < \infty \right\},$$

Obviously,  $M\dot{K}_{p,q(\cdot)}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ .

When  $\lambda = 0$ , we can see that our result below generalize the result in the setting of the Herz space with variable exponent, which proved by Izuki in [17]. So in this paper, we only give the result when  $\lambda > 0$ .

In 2012, Almeida and Drihem<sup>[21]</sup> discuss the boundedness of a wide class of sublinear operators, including maximal, potential and Calderón-Zygmund operators, on variable Herz spaces  $K_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$  and  $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$ . Meanwhile, they also establish Hardy-Littlewood-Sobolev theorems for fractional integrals on variable Herz spaces. In this paper, the author only considers Herz-Morrey space  $M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$  with variable exponent  $q(\cdot)$  but fixed  $\alpha \in \mathbb{R}$  and  $p \in (0, \infty)$ . However, for the case of the exponent  $\alpha(\cdot)$  is variable as well, we can refer to the furthermore work for the author.

Next we state some properties of variable exponent. Cruz-Uribe et al<sup>[4]</sup> and Nekvinda<sup>[10]</sup> proved the following sufficient conditions independently. Moreover, we note that Diening<sup>[6]</sup> proved the following proposition in the case of  $E$  is bounded, and Nekvinda<sup>[10]</sup> gave a more general condition in place of (2).

**Proposition 2.1.** Suppose that  $E$  is an open set, If  $q(\cdot) \in \mathcal{P}(E)$  satisfies the inequality

$$|q(x) - q(y)| \leq \frac{-C}{\ln(|x - y|)} \quad \text{if } |x - y| \leq 1/2, \quad (1)$$

$$|q(x) - q(y)| \leq \frac{C}{\ln(e + |x|)} \quad \text{if } |y| \geq |x|, \quad (2)$$

where  $C > 0$  is a constant independent of  $x$  and  $y$ , then we have  $q(\cdot) \in \mathcal{B}(E)$ .

In order to prove our main theorem, we also need the following result which is the Hardy-Littlewood-Sobolev theorem on Lebesgue spaces with variable exponent due to Capone, Cruz-Uribe and Fiorenza<sup>[22]</sup> (see Theorem 1.8). We remark that this result is initially proved by Diening<sup>[23]</sup> provided that  $q_1(\cdot)$  is constant outside of a large ball.

**Proposition 2.2.** <sup>[22]</sup> Suppose that  $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfies conditions (1) and (2) in Proposition 2.1.  $0 < \beta < n/(q_1)_+$  and define  $q_2(\cdot)$  by

$$\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\beta}{n}.$$

Then we have

$$\|I_\beta f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$

for all  $f \in L^{q_1(\cdot)}(\mathbb{R}^n)$ .

In addition, the following result for the boundedness of  $I_{\beta,b}^m$  on the Lebesgue spaces with variable exponent will be used in the proof of our main theorem.

**Proposition 2.3.** Suppose that  $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfies conditions (1) and (2) in Proposition 2.1. Let  $m \in \mathbb{N}$ ,  $0 < \beta < n/(q_1)_+$ , Define the variable exponent  $q_2(\cdot)$  by

$$\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\beta}{n}.$$

Then  $I_{\beta,b}^m$  is bounded from  $L^{q_1(\cdot)}(\mathbb{R}^n)$  into  $L^{q_2(\cdot)}(\mathbb{R}^n)$  for all  $f \in L^{q_1(\cdot)}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ .

The idea of the proof for Proposition 2.3 comes from the Theorem 1 in [17]. We omit the details.

The next lemma describes the generalized Hölder's inequality and the duality of  $L^{q(\cdot)}(E)$ . The proof is found in [1].

**Lemma 2.1.** <sup>[1]</sup> Suppose that  $q(\cdot) \in \mathcal{P}(E)$ , Then the following statements hold.

1) (generalized Hölder's inequality) For all  $f \in L^{q(\cdot)}(E)$  and all  $g \in L^{q'(\cdot)}(E)$ , we have

$$\int_E |f(x)g(x)|dx \leq r_q \|f\|_{L^{q(\cdot)}(E)} \|g\|_{L^{q'(\cdot)}(E)},$$

where  $r_q = 1 + 1/q_- - 1/q_+$ .

2) For all  $f \in L^{q(\cdot)}(E)$ , we have

$$\|f\|_{L^{q(\cdot)}(E)} \leq \sup \left\{ \int_E |f(x)g(x)|dx : \|g\|_{L^{q'(\cdot)}(E)} \leq 1 \right\}.$$

**Lemma 2.2.** <sup>[19]</sup> If  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then there exists a positive constant  $\delta \in (0, 1)$  and  $C > 0$  such that

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^\delta$$

holds for all balls  $B$  in  $\mathbb{R}^n$  and all measurable subsets  $S \subset B$ .

**Lemma 2.3.** <sup>[19]</sup> If  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then there exists a positive constant  $C > 0$  such that

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C$$

for all balls  $B$  in  $\mathbb{R}^n$ .

**Lemma 2.4.** <sup>[18]</sup> Let  $b \in \text{BMO}(\mathbb{R}^n)$ ,  $m \in \mathbb{N}$ ,  $i, j \in \mathbb{Z}$  with  $i < j$ . Then we have

$$C^{-1} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \leq \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \cdot \chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^m,$$

$$\|(b - b_{B_i})^m \cdot \chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C(j - i)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

The above result is proved by Izuki <sup>[18]</sup>. We remark that Lemma 2.4 is a generalization of well-known properties for BMO spaces.

### 3 Main theorem and its proof

In this section we prove the boundedness for the higher order commutator of fractional integrals on Herz-Morrey spaces with variable exponent under some conditions.

Let  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (1) and (2) in Proposition 2.1. Then so does  $q'(\cdot)$ . In particular, we can see that  $q(\cdot), q'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$  from Proposition 2.1. Therefore applying Lemma 2.2 when  $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ , we can take constant  $\delta_1 \in (0, 1/(q'_1)_+)$ ,  $\delta_2 \in (0, 1/(q_1)_+)$  such that

$$\frac{\|\chi_S\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2} \quad (3)$$

for all balls  $B$  in  $\mathbb{R}^n$  and all measurable subsets  $S \subset B$ .

Our main result can be stated as follows.

**Theorem 3.1.** Suppose that  $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfies conditions (1) and (2) in Proposition 2.1. Define the variable exponent  $q_2(\cdot)$  by

$$\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\beta}{n}.$$

Let  $m \in \mathbb{N}$ ,  $0 < p_1 \leq p_2 < \infty$ ,  $\lambda > 0$ ,  $0 < \beta < n/(q_1)_+$ ,  $\lambda - n\delta_2 < \alpha < \lambda + n\delta_1$ , where  $\delta_1 \in (0, 1/(q'_1)_+)$  and  $\delta_2 \in (0, 1/(q_2)_+)$  are the constants appearing in (3). Then  $I_{\beta,b}^m$  is bounded from  $M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  into  $M\dot{K}_{p_2,q_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  for all  $f \in M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ .

*Proof.* For  $\forall f \in M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  and  $\forall b \in \text{BMO}(\mathbb{R}^n)$ . If we denote  $f_j := f \cdot \chi_j = f \cdot \chi_{A_j}$  for each  $j \in \mathbb{Z}$ , then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

Because of  $0 < p_1/p_2 \leq 1$ , we apply inequality

$$\left( \sum_{i=-\infty}^{\infty} |a_i| \right)^{p_1/p_2} \leq \sum_{i=-\infty}^{\infty} |a_i|^{p_1/p_2},$$

and obtain

$$\begin{aligned} \|I_{\beta,b}^m(f)\|_{M\dot{K}_{p_2,q_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_2} \|I_{\beta,b}^m(f) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_2} \right)^{p_1/p_2} \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \|I_{\beta,b}^m(f) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=k-1}^{k+1} \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=k+2}^{\infty} \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\ &= C(E_1 + E_2 + E_3). \end{aligned}$$

First we estimate  $E_2$ . Using the Proposition 2.3, we have

$$\begin{aligned}
 E_2 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \left( \sum_{j=k-1}^{k+1} \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
 &\leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \left( \sum_{j=k-1}^{k+1} \|f_j \cdot \chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
 &\leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha p_1} \|f_j \cdot \chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
 &= C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1}.
 \end{aligned}$$

For  $E_1$ . Note that when  $x \in A_k, j \leq k-2$ , and  $y \in A_j$ , then  $|x-y| \sim |x|, 2|y| \leq |x|$ . Therefore, using the generalized Hölder's inequality(see 1), Lemma 2.1), we have

$$\begin{aligned}
 |I_{\beta,b}^m(f_j)(x) \cdot \chi_k(x)| &\leq C \int_{A_j} \frac{|f_j(y)| |b(x) - b(y)|^m}{|x-y|^{n-\beta}} dy \cdot \chi_k(x) \\
 &\leq C 2^{k(\beta-n)} \int_{A_j} |f_j(y)| |b(x) - b(y)|^m dy \cdot \chi_k(x) \\
 &\leq C 2^{k(\beta-n)} \left( |b(x) - b_{B_j}|^m \int_{A_j} |f_j(y)| dy + \int_{A_j} |f_j(y)| |b(y) - b_{B_j}|^m dy \right) \cdot \chi_k(x) \\
 &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( |b(x) - b_{B_j}|^m \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \right. \\
 &\quad \left. + \|(b - b_{B_j})^m \chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \right) \cdot \chi_k(x).
 \end{aligned}$$

Thus, from Lemma 2.4, and note that  $\|\chi_i\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leq \|\chi_{B_i}\|_{L^{s(\cdot)}(\mathbb{R}^n)}$ , it follows that

$$\begin{aligned}
 \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( \|(b - b_{B_j})^m \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \right. \\
 &\quad \left. + \|(b - b_{B_j})^m \chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \\
 &\leq C 2^{k(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \right. \\
 &\quad \left. + \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \\
 &\leq C 2^{k(\beta-n)} (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
 \end{aligned} \tag{4}$$

Note that  $\chi_{B_k}(x) \leq C 2^{-k\beta} I_\beta(\chi_{B_k})(x)$  (see page 350, [19]), by Proposition 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}
 \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{-k\beta} \|I_\beta(\chi_{B_k})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
 &\leq C 2^{-k\beta} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
 \end{aligned} \tag{5}$$

Using Lemma 2.2, Lemma 2.3, (3) and (5), we have

$$\begin{aligned}
 2^{k(\beta-n)} \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq 2^{k(\beta-n)} \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \cdot 2^{-k\beta} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
 &\leq C \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \cdot 2^{-kn} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq C \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}^{-1}
 \end{aligned} \tag{6}$$

$$= C \frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \leq C 2^{(j-k)n\delta_1}.$$

On the other hand, note the following fact

$$\begin{aligned} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha} \left( 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\ &\leq 2^{-j\alpha} \left( \sum_{i=-\infty}^j 2^{i\alpha p_1} \|f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \\ &= 2^{j(\lambda-\alpha)} \left( 2^{-j\lambda} \left( \sum_{i=-\infty}^j 2^{i\alpha p_1} \|f_i\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right)^{1/p_1} \right) \\ &\leq C 2^{j(\lambda-\alpha)} \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}. \end{aligned} \tag{7}$$

Thus, combining (4), (6) and (7), and using  $\alpha < \lambda + n\delta_1$ , it follows that

$$\begin{aligned} E_1 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} \|I_{\beta, b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} (k-j)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{-(k-j)n\delta_1} \right)^{p_1} \right) \\ &\leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{p_1} \\ &\quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=-\infty}^{k-2} (k-j)^m 2^{(k-j)(\alpha-\lambda-n\delta_1)} \right)^{p_1} \right) \\ &\leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{p_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \right) \\ &\leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1, q_1(\cdot)}^{\alpha, \lambda}(\mathbb{R}^n)}^{p_1}. \end{aligned}$$

Now, let us turn to estimate for  $E_3$ . Note that when  $x \in A_k, j \geq k+2$ , and  $y \in A_j$ , then  $|x-y| \sim |y|, 2|x| \leq |y|$ . Therefore, using the generalized Hölder's inequality (see 1), Lemma 2.1), we have

$$\begin{aligned} |I_{\beta, b}^m(f_j)(x) \cdot \chi_k(x)| &\leq C \int_{A_j} \frac{|f_j(y)| |b(x) - b(y)|^m}{|x-y|^{n-\beta}} dy \cdot \chi_k(x) \\ &\leq C 2^{j(\beta-n)} \int_{A_j} |f_j(y)| |b(x) - b(y)|^m dy \cdot \chi_k(x) \\ &\leq C 2^{j(\beta-n)} \left( |b(x) - b_{B_k}|^m \int_{A_j} |f_j(y)| dy + \int_{A_j} |f_j(y)| |b(y) - b_{B_k}|^m dy \right) \cdot \chi_k(x) \\ &\leq C 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( |b(x) - b_{B_k}|^m \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} + \|(b - b_{B_k})^m \chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right) \cdot \chi_k(x). \end{aligned}$$

Using Lemma 2.4, it follows that

$$\begin{aligned} \|I_{\beta, b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( \|(b - b_{B_k})^m \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right. \\ &\quad \left. + \|(b - b_{B_k})^m \chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right) \\ &\leq C 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left( \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right. \\ &\quad \left. + \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right) \end{aligned} \tag{8}$$

$$\begin{aligned}
 & + (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big) \\
 & \leq C 2^{j(\beta-n)} (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.
 \end{aligned}$$

Note that  $\chi_{B_j}(x) \leq C 2^{-j\beta} I_\beta(\chi_{B_j})(x)$  (see page 350, [19]), by Proposition 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}
 \|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} & \leq C 2^{-j\beta} \|I_\beta(\chi_{B_j})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
 & \leq C 2^{-j\beta} \|\chi_{B_j}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
 & \leq C 2^{-j\beta} 2^{jn} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}^{-1}.
 \end{aligned}$$

Thus, we have

$$2^{j(\beta-n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq C \|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{-1}. \quad (9)$$

Using Lemma 2.2, Lemma 2.3, (3) and (9), we have

$$\begin{aligned}
 2^{j(\beta-n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} & \leq C \|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{-1} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
 & \leq C \frac{\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \\
 & \leq C 2^{(k-j)n\delta_2}.
 \end{aligned} \quad (10)$$

Thus, combining (7), (8) and (10), and using  $\lambda - n\delta_2 < \alpha$ , it follows that

$$\begin{aligned}
 E_3 & = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=k+2}^{\infty} \|I_{\beta,b}^m(f_j) \cdot \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \right) \\
 & \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \left( \sum_{j=k+2}^{\infty} (j-k)^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{-(j-k)n\delta_2} \right)^{p_1} \right) \\
 & \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1} \\
 & \quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=k+2}^{\infty} (j-k)^m 2^{(j-k)(\lambda-\alpha-n\delta_2)} \right)^{p_1} \right) \\
 & \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \right) \\
 & \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp_1} \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{p_1}.
 \end{aligned}$$

This finishes the proof of Theorem 3.1.  $\square$

When  $\lambda = 0$ , our main result also hold on Herz space with variable exponent, and generalize the result of Izuki<sup>[17]</sup> (see Theorem 3). When  $m = 0$ , we also improve the result for Izuki<sup>[19]</sup> (see Theorem 2).

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